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Quantum mechanics from a geometric-observer's viewpoint

Bozhidar Z Iliev†

Department Mathematical Modeling, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria

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Abstract. We propose a version of non-relativistic quantum mechanics in which the pure states of a quantum system are described as sections of a Hilbert (generally infinitely-dimensional) fibre bundle over spacetime. Evolution is governed via a (kind of) parallel transport in this bundle. Some problems concerning observables are considered. The equations of motion for the state sections and observables are derived. We show that up to a constant the matrix of the coefficients of the evolution operator (transport) coincides with the matrix of the Hamiltonian of the investigated quantum system.

1. Introduction

In conventional non-relativistic quantum mechanics a pure state of some quantum system is described by a state vector in a (generic infinitely-dimensional) Hilbert space [1, 2]. The time evolution of this vector is governed by the Schrödinger equation but, for some purposes, it can also be represented (equivalently) via the so-called evolution operator [2]. In [3] (see also [4] which is almost a review of [3], but also contains new material) an interpretation of this operator as a parallel transport in a (generic infinitely dimensional) vector bundle over spacetime is suggested.

Regardless of the fact that [3, 4] do not meet any present-day mathematical standards of rigor, they do contain some interesting ideas which we develop in the present work. On the one hand, we accept the description of a quantum evolution as a (parallel) transport (of sections) in a (Hilbert) fibre bundle over spacetime. On the other, we agree that quantities like the state vectors should generally explicitly depend on the observer with respect to which they are referred, a fact which is usually implicitly assumed. An analogous feature can also be found in Prugovečki's approach to quantum theory (see [5] for a selective summary), but we shall not deal with it here. In the present work we apply these ideas to the description of pure quantum states.

This paper develops some aspects of the geometric approach to non-relativistic quantum mechanics based on the Schrödinger equation. We make an attempt to apply the theory of fibre bundles (and (linear) transports on them) to quantum mechanics. In particular, we describe the time evolution of pure quantum states, conventionally governed by the Schrödinger equation, as a linear transport of sections (of a fibre bundle over spacetime) along the trajectory (world line) of a given (local, i.e. point-like) observer. It should be noted that this transport is not in the 'direction of time', it is along the observer's world line parametrized with the (observer's proper) time. By means of the transport, we transform

† E-mail address: bozho@inrne.acad.bg

section values from one spacetime point to another. This 'transportation' may be towards increasing as well as decreasing time values, which reflects the fact that the Schrödinger equation (together with certain initial condition(s)) predicts wavefunction values in the future as well as in the past.

In section 2 we briefly review the notion of a linear, in particular parallel, transport along paths in vector bundles. In section 3 we make our basic assumptions concerning the geometry of quantum mechanics. We suppose the (pure) states of a quantum system to be described by sections of a Hilbert fibre bundle whose standard fibre is a Hilbert space, isomorphic to that one of the conventional approach. This bundle is assumed to be endowed with a (Hermitian fibre) metric by means of which the expectation (mean) values of the observables are determined. The time evolution of a system's state is governed by a (kind of) parallel transport found via the Schrödinger equation. This transport is supposed to preserve the scalar products defined by the metric. In our approach the observables are represented as bundle morphisms. In section 4 we investigate certain consequences of the natural requirement that the expectation values must be independent of the additional path via which they are defined at points different from that at which the observer is situated. Section 5 is devoted to the equations of motion governing the time evolution of the state sections and observables. A remarkable result here is that up to a constant the matrix of the Hamiltonian coincides with the matrix of the coefficients of the evolution operator. In this sense we can state that in our approach the Hamiltonian plays the role of a gauge field. We close the paper with some remarks in section 6.

2. Mathematical preliminaries

In this section we recall some facts concerning linear transport along paths in vector bundles [6].

Let (E, π, M) be a complex vector fibre bundle with base M, total space E, and projection $\pi : E \to M$. The fibres $E_x := \pi^{-1}(x) \subset E$, $x \in M$, are isomorphic vector spaces, i.e. there exists a vector space $\mathcal{H}$ and isomorphisms l_x, $x \in M$ such that $l_x : E_x \to \mathcal{H}$. We do not make any assumptions on the dimensionality of (E, π, M), i.e. $\mathcal{H}$ can have a finite as well as infinite dimension. (Note that the results of [6, 7] cited are also valid in the infinite-dimensional case regardless of the fact that they are proved under the assumption of finite dimensionality.)

By J and $\gamma : J \to M$ we denote a real interval and a path in M, respectively.

A $\mathbb{C}$-*linear transport* (*L-transport*) *along paths* in (E, π, M) is a map $L : \gamma \mapsto L^\gamma$, where $L^\gamma : (s, t) \mapsto L^\gamma_{s\to t}$, $s, t \in J$ is the (L-)transport along γ, and $L^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t))$, called (L-)transport along γ from s to t, satisfies the equalities

$$L^\gamma_{t\to r} \circ L^\gamma_{s\to t} = L^\gamma_{s\to r} \qquad r, s, t \in J \tag{2.1}$$

$$L^\gamma_{s\to s} = \mathrm{id}_{\pi^{-1}(\gamma(s))} \qquad s \in J \tag{2.2}$$

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t} u + \mu L^\gamma_{s\to t} v \qquad \mu, \lambda \in \mathbb{C}, u, v \in \pi^{-1}(\gamma(s)). \tag{2.3}$$

Here id_N denotes the identity map of a set N. The general form of $L^\gamma_{s\to t}$ is described by

$$L^\gamma_{s\to t} = (F^\gamma_t)^{-1} \circ F^\gamma_s \qquad s, t \in J \tag{2.4}$$

with $F^\gamma_s : \pi^{-1}(\gamma(s)) \to Q$, $s \in J$, being one-to-one (linear) maps onto one and the same (complex) vector space Q.

From (2.1) and (2.2) we see that

$$(L^\gamma_{s\to t})^{-1} = L^\gamma_{t\to s}. \tag{2.5}$$

According to [8, theorem 3.1] the set of (resp. linear) transports which are diffeomorphisms and satisfy the locality and reparametrization conditions, i.e. $L^{\gamma}_{s\to t} \in \mathrm{Diff}(\pi^{-1}(\gamma(s)), \pi^{-1}(\gamma(t)))$, $L^{\gamma|J'}_{s\to t} = L^{\gamma}_{s\to t}$ for $s, t \in J'$, with J' being a subinterval of J, and $L^{\gamma\circ\tau}_{s\to t} = L^{\gamma}_{\tau(s)\to\tau(t)}$, $s, t \in J''$ with τ being a 1:1 map of an $\mathbb{R}$-interval J'' onto J, are in one-to-one correspondence with the (axiomatically defined (resp. linear)) parallel transports (along curves). So, the usual parallel transport along γ from $\gamma(s)$ to $\gamma(t)$, assigned to a linear connection, is a standard realization of the general (resp. linear) transport $L^{\gamma}_{s\to t}$.

Let g be a (Hermitian) fibre metric on (E, π, M), i.e. [9] $g : x \mapsto g_x$ with $g_x : E_x \times E_x \to \mathbb{C}$, $x \in M$, being non-degenerate Hermitian forms, i.e. g_x are Hermitian, nondegenerate maps which are $\mathbb{C}$-linear in the second argument and $\mathbb{C}$-antilinear in the first one. A fibre metric g and an L-transport L are called *consistent* (respectively along γ) if L preserves the scalar product defined by g, i.e. [7]

$$g_{\gamma(s)} = g_{\gamma(t)} \circ (L^{\gamma}_{s\to t} \times L^{\gamma}_{s\to t}) \qquad s, t \in J \tag{2.6}$$

for all (respectively the given) γ. Different results concerning this consistency can be found in [7].

If $h : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a Hermitian, nondegenerate map which is $\mathbb{C}$-antilinear in the first argument and $\mathbb{C}$-linear in the second one (a Hermitian metric (scalar product) in $\mathcal{H}$), then, evidently, the map $g : x \mapsto g_x := h(l_x\cdot, l_x\cdot) : E_x \times E_x \to \mathbb{C}$ is a fibre metric on (E, π, M). Conversely, if g is a fibre metric in (E, π, M) then, using the results from [7], it can easily be proved that the map $h := g_x(l_x^{-1}\cdot, l_x^{-1}\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a Hermitian metric on $\mathcal{H}$ iff there is an L-transport along paths consistent with g†.

Let $\eta : J \times J' \to M$ be a C^2 map. The curvature operator $\mathcal{R}^{\eta}(s, t) : E_{\eta(s,t)} \to E_{\eta(s,t)}$ of the L-transport L with respect to η at $(s, t) \in J \times J'$ is defined by [10, equation (3.1)].

Let $\delta, \epsilon \in \mathbb{R}_+$ be such that $(s + \delta, t + \epsilon) \in J \times J'$ and λ be the (oriented) closed path defined as a product of the following paths: $\sigma \mapsto \eta(s + \sigma, t)$ for $\sigma \in [0, \delta]$, $\tau \mapsto \eta(s + \delta, t + \tau)$ for $\tau \in [0, \epsilon]$, $\sigma \mapsto \eta(s + \delta - \sigma, t + \epsilon)$ for $\sigma \in [0, \delta]$, and $\tau \mapsto \eta(s, t + \epsilon - \tau)$ for $\tau \in [0, \epsilon]$. Hence λ is a closed (oriented) loop connecting the points $\eta(s, t)$, $\eta(s + \delta, t)$, $\eta(s + \delta, t + \epsilon)$, $\eta(s, t + \epsilon)$, and $\eta(s, t)$ in the written order.

Supposing $L^{\gamma}_{s\to t}$ to have a C^2 dependence on s (and thereof on t) and using [6, proposition 2.1], we obtain, after some calculations, that the composition of the successive L-transports of a vector at $\eta(s, t)$ along the paths forming λ is represented by an operator whose matrix has the following expansion (see [11, section 4])

$$\mathbb{1} - \delta\epsilon\boldsymbol{\mathcal{R}}^{\eta}(s, t) + O(\delta^3) + O(\epsilon^3) + O(\delta^2\epsilon) + O(\epsilon^2\delta) \tag{2.7}$$

in some field of local bases. Here $\mathbb{1}$ is the unit matrix and $\boldsymbol{\mathcal{R}}^{\eta}(s, t)$ is the matrix corresponding to $\mathcal{R}^{\eta}(s, t)$. If the L-transport along a product of paths is equal to the composition of the L-transports along the corresponding paths of the product (in the respective order), then this operator coincides with the linear transport along λ.

3. Basic differential-geometric assumptions

The state of a quantum system will be described by a quantity ψ assumed to be a section of a vector bundle (E, π, M) over the spacetime M: $\psi \in \mathrm{Sec}(E, \pi, M) := \{\xi : \xi : M \to E, \pi\circ\xi = \mathrm{id}_M\}$. The bundle (E, π, M) is not supposed to be locally trivial. The typical fibre

† Notice that h is always independent of $x \in M$. The transition $g \leftrightarrow h$ is similar to the one in (gauge) gravitational theories, where (at a fixed point) one transforms a general point-depending metric to the Minkowski metric and *vice versa*, or, equivalently, to the transition from a general basis to a local fierbein.

$\mathcal{H}$ is supposed to be a Hilbert space, so such are all (isomorphic to $\mathcal{H}$) fibres $E_x := \pi^{-1}(x)$, $x \in M$.

One can associate an L-transport along paths with the evolution of any non-relativistic quantum system. For pure states this can be done as follows (cf [3]). Let $\gamma : J \to M$ be the world line of an observer B. We interpret $t \in J$ as a proper time (eigentime) of B. We suppose a quantum system to be described by B at $\gamma(t) \in M$, at the 'moment' $t \in J$, by the state vector $\psi_\gamma(t) \in E_{\gamma(t)}$, generally depending on γ and t separately; in particular, it may depend only on $\gamma(t)$. Let B describe the evolution of the system with a Hamiltonian $H_\gamma(t)$ through the Schrödinger equation, which in matrix form reads†

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}_\gamma(t) = \boldsymbol{H}_\gamma(t)\boldsymbol{\psi}_\gamma(t). \tag{3.1}$$

Here and from now on in our text we denote with bold symbols the matrices corresponding to vectors or operators in (a) given (field of) bases (for details about infinite-dimensional matrices, see, e.g., [12]). We can write

$$\psi_\gamma(t) = U_\gamma(t, t_0)\psi_\gamma(t_0) \qquad t, t_0 \in J \tag{3.2}$$

where $t_0 \in J$ is fixed and $U_\gamma(t, t_0)$ is a linear operator, called the *evolution operator*, defined as the unique solution of the initial-value problem [2]

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\boldsymbol{U}_\gamma(t, t_0) = \boldsymbol{H}_\gamma(t)\boldsymbol{U}_\gamma(t, t_0) \tag{3.3}$$

$$U_\gamma(t_0, t_0) = \mathrm{id}_{E_{\gamma(t_0)}}. \tag{3.4}$$

It is evident that $U_\gamma(t, t_0) : E_{\gamma(t_0)} \to E_{\gamma(t)}$ is an L-transport along γ from t_0 to t, i.e. $U : \gamma \mapsto U_\gamma : (t, t_0) \mapsto U_\gamma(t, t_0)$ is an L-transport along paths in (E, π, M). Moreover, under certain natural assumptions (cf [3]), U turns to be a (usual) parallel transport.

The fibre bundle (E, π, M) is assumed to be endowed with two structures: a linear transport along paths L, which is supposed to coincide with the above-defined evolution operator U‡ and a Hermitian fibre metric g consistent with it. For brevity, as usual, we use the bracket notation:

$$\langle\psi(x)|\xi(x)\rangle_x := g_x(\psi(x), \xi(x)) \qquad x \in M, \psi, \xi \in \mathrm{Sec}(E, \pi, M). \tag{3.5}$$

So, now the consistency condition (2.6) reads

$$\langle\psi(\gamma(s))|\xi(\gamma(s))\rangle_{\gamma(s)} = \langle L^\gamma_{s\to t}\psi(\gamma(s))|L^\gamma_{s\to t}\xi(\gamma(s))\rangle_{\gamma(t)}. \tag{3.6}$$

Equation (3.6) restricts us to consider only *unitary* L-transports with respect to the metric. In fact, if we define the *Hermitian conjugate* to $L^\gamma_{s\to t}$ transport, ${}^\dagger L^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t))$ by

$$\langle L^\gamma_{s\to t}\psi(\gamma(s))|\xi(\gamma(t))\rangle_{\gamma(t)} =: \langle\psi(\gamma(s))|{}^\dagger L^\gamma_{t\to s}\xi(\gamma(t))\rangle_{\gamma(s)}$$

then, due to (2.5), we see (3.6) to be equivalent to ${}^\dagger L^\gamma_{s\to t} = L^\gamma_{t\to s} = (L^\gamma_{s\to t})^{-1}$§.

Let $\mathcal{O}$ be the set of observables. Its connection with the spacetime is described by a map $\varphi : \mathcal{O} \to \mathrm{Morf}(E, \pi, M)$ assigning to $A \in \mathcal{O}$ a morphism $A_\varphi : E \to E$, i.e. $\pi \circ A_\varphi = \pi$ (and hence $A_\varphi : E_x \to E_x$).

† In this work we present in matrix form all relations containing derivatives. In this way we avoid problems connected with the differentiation of fields of objects defined (or acting) on E; e.g. $\partial\psi_\gamma(t)/\partial t$ is not ('well') defined at all. The invariant form of these relations will be given elsewhere.

‡ Later we preserve the notation L as most of the results hold mathematically for generic L-transport L, not only for the evolution operator U.

§ Dropping the arguments, if $\boldsymbol{U}$ and $\boldsymbol{G}$ are the matrices of the transport and metric, respectively, the last equality is equivalent to ${}^\dagger\boldsymbol{U} = \boldsymbol{G}^{-1}\overline{\boldsymbol{U}}^\top\boldsymbol{G}$.

The set of observers $\mathbf{B}$ consists of maps $B_x : \mathrm{Sec}(E, \pi, M) \to E_x$, observers at x, assigning to any state section ψ a state vector at $x \in M$, i.e. $B_x : \psi \mapsto \psi_B(x)$.

We define the *expectation value* of $A \in \mathcal{O}$ with respect to B_x, when the system has a state section ψ, by

$$\langle A \rangle_{B_x} := \frac{\langle \psi_B(x) | A_\varphi \psi_B(x) \rangle_x}{\langle \psi_B(x) | \psi_B(x) \rangle_x}. \tag{3.7}$$

The vector

$$\psi^\gamma_{B,s,t} := L^\gamma_{s\to t} \psi_B(\gamma(s))$$

can be interpreted as a state vector of the quantum system at $y = \gamma(t)$ 'predicted' by an observer B_x situated at $x = \gamma(s)$. (Here γ may not be the observer's world line.) By definition the expectation value of $A \in \mathcal{O}$ at $y = \gamma(t)$ with respect to B_x, $x = \gamma(s)$, along γ is

$$\langle A \rangle^\gamma_{B,s,t} := \frac{\langle \psi^\gamma_{B,s,t} | A_\varphi \psi^\gamma_{B,s,t} \rangle_{\gamma(t)}}{\langle \psi^\gamma_{B,s,t} | \psi^\gamma_{B,s,t} \rangle_{\gamma(t)}} = \frac{\langle \psi_B(x) | L^\gamma_{t\to s} \circ A_\varphi \circ L^\gamma_{s\to t} \psi_B(x) \rangle_x}{\langle \psi_B(x) | \psi_B(x) \rangle_x} \tag{3.8}$$

where (3.6) was used. Evidently, we have $\langle A \rangle^\gamma_{B,s,s} = \langle A \rangle_{B_{\gamma(s)}}$.

4. Observables and the evolution operator

We assume the expectation value of $A \in \mathcal{O}$ at $y = \gamma(t)$ with respect to an observer B_x to be independent of the path via which it is determined, i.e. for $\beta : J' \to M$ and $\sigma, \tau \in J'$, we demand

$$\langle A \rangle^\gamma_{B,s,t} = \langle A \rangle^\beta_{B,\sigma,\tau} \qquad \text{if } \beta(\sigma) = \gamma(s) \text{ and } \beta(\tau) = \gamma(t). \tag{4.1}$$

This equality is a partial realization of the physical requirement that the observed (expectation) values of the dynamical variables must be independent of the way they are calculated.

For a path $\alpha : J'' \to M$ containing a closed loop at x, i.e. $\alpha(s) = \alpha(t) = x$, for some $s, t \in J''$, this condition reduces to $\langle A \rangle^\alpha_{B,s,t} = \langle A \rangle_{B_x}$ as we can choose β to be $\beta_\sigma : [\sigma, \sigma] \to \{x\}$. Using (3.8) we can rewrite the last condition as $\langle \psi_B(x) | A_\varphi \psi_B(x) \rangle_x = \langle \psi_B(x) | L^\gamma_{t\to s} \circ A_\varphi \circ L^\gamma_{s\to t} \psi_B(x) \rangle_x$. Admitting this equality to be valid for every $\psi_B(x) \in E_x$, $x = \gamma(s)$, we get

$$[L^\alpha_{s\to t}, A_\varphi] = 0 \qquad \text{for any } \alpha \text{ for which } \alpha(s) = \alpha(t) \tag{4.2}$$

where $[\cdot, \cdot]$ denotes the commutator of the corresponding operators. This result is a special case of the equation

$$[L^\gamma_{s\to t} \circ L^\beta_{\sigma\to\tau}, A_\varphi] = 0 \qquad \text{for } \gamma(s) = \beta(\tau) \text{ and } \gamma(t) = \beta(\sigma) \tag{4.3}$$

which is a corollary of (3.8) and (4.1).

In particular, A_φ commutes with the L-transport along any closed path (loop) α. Hence, if we choose $\alpha = \lambda$, with λ being the oriented closed path defined at the end of section 2, then for any L-transport satisfying the condition at the end of section 2, we obtain

$$[\mathcal{R}^\eta(s, t), A_\varphi] = 0 \tag{4.4}$$

where (2.7) was used, i.e. the curvature operator of the mentioned linear transport commutes with all observables. This is a necessary condition for the validity of (4.1).

As we have seen above, the map $L^\alpha_{s\to t}$ for $\alpha(s) = \alpha(t)$ is independent of any local coordinates or trivializations (if any), it generally non-trivially transforms state onto state, and leaves the observables invariant. Consequently it acts and can be considered as a local symmetry transformation.

The linear transport L induces along $\gamma : J \to M$ the following transformation of an observable A_φ, or, more precisely, of $A_\varphi|_{E_{\gamma(t)}}$:

$$A_\varphi \mapsto A_\varphi^\gamma(s,t) := L_{t\to s}^\gamma \circ A_\varphi \circ L_{s\to t}^\gamma : E_{\gamma(s)} \to E_{\gamma(s)}. \tag{4.5}$$

In fact, $A_\varphi^\gamma(s,t)$ is the result of 'L-transportation' of $A_\varphi|_{E_{\gamma(t)}}$ from t to s along γ. Rigorously speaking, the map $A_\varphi|_{E_{\gamma(t)}} \to A_\varphi^\gamma(s,t)$ is a linear transport along γ from t to s in the fibre bundle of bundle morphisms over (E,π,M) (for details, see [13, section 3]).

For the closed path λ and special L-transports defined at the end of section 2 we can substitute (2.7) into (4.5). This gives

$$\boldsymbol{A}_\varphi^\lambda(s,t) = \boldsymbol{A}_\varphi|_{E_{\lambda(s,t)}} + \delta\epsilon[\boldsymbol{\mathcal{R}}^\eta(s,t), \boldsymbol{A}_\varphi] + O((\delta,\epsilon)^3)$$

where $O((\delta,\epsilon)^3)$ means third-order quantities in δ and ϵ. Combining this with (4.4), we find

$$\boldsymbol{A}_\varphi^\lambda(s,t) = \boldsymbol{A}_\varphi|_{E_{\lambda(s,t)}} + O((\delta,\epsilon)^3). \tag{4.6}$$

Substituting equation (4.5) into (3.8), we get

$$\langle A\rangle_{B,s,t}^\gamma := \frac{\langle\psi_B(x)|A_\varphi^\gamma(s,t)\psi_B(x)\rangle_x}{\langle\psi_B(x)|\psi_B(x)\rangle_x}. \tag{4.7}$$

Due to (3.6), (3.8), and (4.1), we evidently have

$$\langle\psi_B(x)|A_\varphi^\gamma(s,t)\psi_B(x)\rangle_x = \langle\psi_{B,s,t}^\gamma|A_\varphi\psi_{B,s,t}^\gamma\rangle_x = \langle\psi_B(x)|A_\varphi^\beta(s',t')\psi_B(x)\rangle_x$$
$$\beta(s') = \gamma(s) = x,\ \beta(t') = \gamma(t).$$

If $r, r', s, t \in J$, then $L_{s\to t}^\gamma = L_{r'\to t}^\gamma \circ L_{r\to r'}^\gamma \circ L_{s\to r}^\gamma$ (see (2.1)). Inserting this equality into (4.5) and using (2.5) we, after some algebra, obtain

$$A_\varphi^\gamma(r,s) \circ L_{r'\to r}^\gamma = L_{r'\to r}^\gamma \circ A_\varphi^\gamma(r',t) : E_{\gamma(r')} \to E_{\gamma(r)}. \tag{4.8}$$

If L^γ is a parallel transport along γ, then putting here $\gamma = \beta^{-1}\alpha\beta$, where $\beta : [a,b] \to M$, $\beta(a) = \gamma(r) = \gamma(r')$, $\beta(b) = \gamma(s) = \gamma(t)$, and $\alpha : [a',b'] \to M$, $\alpha(a') = \alpha(b')$, we get

$$[L_{a'\to b'}^\alpha, A_\varphi^\beta(a,b)] = 0$$

for every closed path α located at y and any path β containing y and x. However, for general L-transports this equality may not hold.

Let us assume that for the point $x \in M$ there is a neighbourhood $U \ni x$ such that x can be connected by a path with any point from U. Then there is a homotopy $\beta : U \times [0,1] \to M$ connecting $\chi_x : U \to x$ and the inclusion map $\imath_U : U \to M, \imath_U(y) = y \in U$, i.e. $\beta(\cdot,0) := \chi_x$ and $\beta(\cdot,1) := \imath_U$. Hence, the expectation value of $A \in \mathcal{O}$ at any $y \in U$ with respect to an observer B_x is

$$\langle A\rangle_B^{\beta(y,\cdot)} := \langle A\rangle_{B,0,1}^{\beta(y,\cdot)} = \frac{\langle\psi_B(x)|A_\varphi^{\beta(y,\cdot)}\psi_B(x)\rangle_x}{\langle\psi_B(x)|\psi_B(x)\rangle_x}$$

where $A_\varphi^{\beta(y,\cdot)} := A_\varphi^{\beta(y,\cdot)}(0,1) = L_{1\to 0}^{\beta(y,\cdot)} \circ A_\varphi \circ L_{0\to 1}^{\beta(y,\cdot)} : E_x \to E_x$.

Every intermediate point $\beta(y,\tau)$, $\tau \in [0,1]$ is connected with x (besides via $\beta(y,\cdot)$) also by the path $\beta_{y,\tau} := \beta(y,\cdot)|_{[0,\tau]} : t \mapsto \beta(y,t)$ for $t \in [0,\tau]$. We have

$$\langle A\rangle_B^{\beta_{y,\tau}} := A_{B,0,\tau}^{\beta_{y,\tau}} = \frac{\langle\psi_B(x)|A_\varphi^{\beta_{y,\tau}}\psi_B(x)\rangle_x}{\langle\psi_B(x)|\psi_B(x)\rangle_x}$$

with

$$A_{\varphi}^{\beta_{y,\tau}} := A_{\varphi}^{\beta_{y,\tau}}(0,\tau) = L_{\tau\to 0}^{\beta_{y,\tau}} \circ A_{\varphi} \circ L_{0\to\tau}^{\beta_{y,\tau}} : E_x \to E_x. \tag{4.9}$$

Let us assume that the evolution of a quantum system along $\beta_{y,\tau}$ is given by $\psi_{\beta_{y,\tau}}(t) = L_{0\to t}^{\beta_{y,\tau}}\psi_{\beta_{y,\tau}}(0)$ through the Schrödinger equation (3.1), i.e. the L-transport satisfies equation (3.3):

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\boldsymbol{L}_{0\to t}^{\beta_{y,\tau}} = \boldsymbol{H}_{\beta_{y,\tau}}(t)\boldsymbol{L}_{0\to t}^{\beta_{y,\tau}} \qquad \text{so that} \quad \mathrm{i}\hbar\frac{\partial}{\partial t}\boldsymbol{L}_{t\to 0}^{\beta_{y,\tau}} = -\boldsymbol{L}_{t\to 0}^{\beta_{y,\tau}}\boldsymbol{H}_{\beta_{y,\tau}}(t).$$

Differentiating the matrix form of (4.9) with respect to τ and using these equalities, we get

$$\mathrm{i}\hbar\frac{\partial}{\partial\tau}\boldsymbol{A}_{\varphi}^{\beta_{y,\tau}} = -[\boldsymbol{H}_{\beta_{y,\tau}}^{\beta_{y,\tau}}(\tau), \boldsymbol{A}_{\varphi}^{\beta_{y,\tau}}] \tag{4.10}$$

where $H_{\beta_{y,\tau}}^{\beta_{y,\tau}}(\tau) := L_{\tau\to 0}^{\beta_{y,\tau}} \circ H_{\beta_{y,\tau}}(\tau) \circ L_{0\to\tau}^{\beta_{y,\tau}}$ is the bundle morphism restricted on E_x corresponding to the Hamiltonian $H_{\beta_{y,\tau}}(\tau)$ according to (4.9).

5. Equations of motion

The Schrödinger equation (3.1) is an equation of motion for the state vectors. Equation (4.10) plays the same role with respect to observables. Below we consider briefly the analogues of these equations in the theory considered here with linear transports.

Let $B \in \mathbf{B}$ be an observer with a world line $\gamma : J \to M$, i.e. $B : x \mapsto B_x : \mathrm{Sec}(E,\pi,M) \to E_x, x = \gamma(s), s \in J$. Let for a fixed $s_0 \in J$ the state vector of the quantum system be $\psi_0 := \psi_\gamma(s_0) \in E_{\gamma(s_0)}$. We assume that along γ the state vector with respect to B at $\gamma(s)$, $s \in J$, is obtained via some linear transport L along paths, viz

$$\psi_\gamma(s) = L_{s_0\to s}^{\gamma}\psi_0. \tag{5.1}$$

This equation is our analogue of (3.2) and it plays the role of the state vector (section) equation of motion.

Let us define the matrix $\boldsymbol{\Gamma}_\gamma(s)$ of the *coefficients* of an L-transport by

$$\boldsymbol{\Gamma}_\gamma(s) := \left(\frac{\partial}{\partial s}\boldsymbol{L}_{s\to t}^{\gamma}\right)_{t=s}. \tag{5.2}$$

Evidently (see (2.4))

$$\boldsymbol{\Gamma}_\gamma(s) = -\left(\frac{\partial}{\partial t}\boldsymbol{L}_{s\to t}^{\gamma}\right)_{t=s} = (\boldsymbol{F}_s^{\gamma})^{-1}\frac{\partial\boldsymbol{F}_s^{\gamma}}{\partial s}. \tag{5.3}$$

Now we shall prove that *up to a constant in our theory* $\boldsymbol{\Gamma}_\gamma(s)$ *plays the role of a (matrix) Hamiltonian* describing the system's evolution through the Schrödinger-type equation. In fact, from (5.2) and (2.4) we find

$$\frac{\partial}{\partial t}\boldsymbol{L}_{s\to t}^{\gamma} = -\boldsymbol{\Gamma}_\gamma(t)\boldsymbol{L}_{s\to t}^{\gamma}. \tag{5.4}$$

Combining this equation with (5.1), we confirm ourselves that $\psi_\gamma(t)$ satisfies the Schrödinger equation (3.1) with $\boldsymbol{H}_\gamma(t) = -\mathrm{i}\hbar\boldsymbol{\Gamma}_\gamma(t)$, which proves our assertion.

If the system evolution is described by a Hamiltonian $H_\gamma(t)$ via (3.1), then our results hold for $\boldsymbol{\Gamma}_\gamma(t) = -\boldsymbol{H}_\gamma(t)/\mathrm{i}\hbar$.

If $\boldsymbol{\Gamma}_\gamma(s)$ is a given operator, then equation (5.4) with an initial condition (2.2) uniquely defines the linear transport L.

The matrix $\boldsymbol{\Gamma}_\gamma(t)$ can also be called a 'gauge matrix' as it defines the 'extended (covariant) derivatives'. In fact, recalling [6] that the differentiation along paths $\mathcal{D} : \gamma \mapsto \mathcal{D}^\gamma$ defined by L acts on a C^1 section ψ according to

$$(\mathcal{D}^\gamma \psi)(s) = \mathcal{D}_s^\gamma \psi = \left[\frac{\partial}{\partial \epsilon} (L^\gamma_{s+\epsilon \to s} \psi(\gamma(s+\epsilon))) \right] \Bigg|_{\epsilon = 0}$$

we see that $\mathcal{D}_s^\gamma : \mathrm{Sec}^1(E, \pi, M) \to \pi^{-1}(\gamma(s))$ and the matrix of the components of $\mathcal{D}_s^\gamma \psi$ is $\partial \boldsymbol{\psi}(s)/\partial s + \boldsymbol{\Gamma}_\gamma(s)\boldsymbol{\psi}(s)$.

The above discussion allows us to interpret the usual Hamiltonian as a gauge operator, or, in some sense, as a 'generalized affine connection' along paths.

Now to derive the generalization of (4.10) we have to differentiate the matrix form of (4.5) with respect to s and use (5.4). Thus we get

$$\frac{\partial}{\partial s} \boldsymbol{A}^\gamma_\varphi(s, t) = -[\boldsymbol{\Gamma}_\gamma(s), \boldsymbol{A}^\gamma_\varphi(s, t)]. \tag{5.5}$$

This is the equation of motion required for the observables. In terms of the Hamiltonian, because of $\boldsymbol{\Gamma}_{\gamma(t)} = -\boldsymbol{H}_\gamma(t)/\mathrm{i}\hbar$, it reads

$$\mathrm{i}\hbar \frac{\partial}{\partial s} \boldsymbol{A}^\gamma_\varphi(s, t) = [\boldsymbol{H}_\gamma(s), \boldsymbol{A}^\gamma_\varphi(s, t)]. \tag{5.6}$$

Analogously, differentiating the matrix form of (4.5) with respect to t, we find

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \boldsymbol{A}^\gamma_\varphi(s, t) = -[\boldsymbol{H}_\gamma(s, t), \boldsymbol{A}^\gamma_\varphi(s, t)] \tag{5.7}$$

where $\boldsymbol{\Gamma}_\gamma(t) = -\boldsymbol{H}_\gamma(t)/\mathrm{i}\hbar$ was used and $H_\gamma(s, t) := L^\gamma_{t\to s} \circ H_\gamma(t) \circ L^\gamma_{s \to t}$ is the morphism restricted on $E_{\gamma(s)}$ corresponding to the Hamiltonian H_γ. Equation (5.7) is an evident generalization of (4.10) for arbitrary path γ.

6. Conclusion

The approach to non-relativistic quantum mechanics developed in this paper is intended to bring it to the class of physical theories mathematically based on the formalism of fibre bundles. At the present level, the new approach is equivalent to the conventional one which will be proved in detail elsewhere.

The novel 'bundle' treatment of old problems reveals new possibilities for generalizations and interpretations (cf the similar advantages of Prugovečki's theory [5]). In particular, it is likely that the bundle formalism in quantum theory will be useful for the unification of quantum mechanics and gravitation. A reason for this hope is the fact that we have not used any concrete model of spacetime; it can be flat as well as curved and, generally, has to be determined by another theory such as special or general relativity.

The fibre bundle formalism also seems applicable to relativistic quantum theory and field theory which will be a subject of other works. Since the purpose of the present paper is a geometric description of the non-relativistic case, here we want only to make some comments on these items.

The fibre bundle approach to relativistic quantum mechanics, generally, needs a different mathematical base than the one used in this work. A typical example of this kind is a special-relativistic particle described by the Klein–Gordon equation. An essential point here is that this is a second-order partial differential equation with respect to time. This implies that an initial value of the wavefunction is not sufficient for the unique determination of its other values; for this one needs the initial values of the wavefunction and its first time derivative.

So, we cannot directly apply a 'linear transportation' to obtain wavefunction values from one another (for details, see [14, section 5]). A way to overcome this problem is to consider a fibre bundle, the elements of whose fibres have two components formed from the wave function ψ and its first time derivative $\partial\psi/\partial t$, i.e. they are of the type $(\psi, \partial\psi/\partial t)^\top$. Such a two-component wavefunction satisfies a first-order partial differential equation with respect to time [1, ch XX, section 5]. This last equation admits consideration analogous to that of the Schrödinger equation presented in this paper. The above-mentioned difficulty does not arise for particles described via the Dirac equation. In fact, since the Dirac equation can be written as [1, ch XX, section 6] $\mathrm{i}\hbar\partial\psi/\partial t = H_\mathrm{D}\psi$, H_D being the Dirac Hamiltonian, we can apply *mutatis mutandis* the present investigation to Dirac particles. For this purpose we have to replace the non-relativistic Hamiltonian with the Dirac Hamiltonian, the Hilbert space with the space of 4-spinors, etc.

In connection with further applications of the bundle approach to quantum field theory, we note the following. Since in this theory the matter fields are represented by operators acting on (wave) functions from some space, the matter fields in their bundle modification should be described via morphisms of a suitable fibre bundle whose sections will represent the (wave) function. We can also, equivalently, say that in this way the matter fields would be sections of the fibre bundle of bundle morphisms of the mentioned suitable bundle. An important point here is that the matter fields are primarily related to the bundle arising over the spacetime, not to the spacetime itself to which are directly related other structures, such as connections and the principle bundle over it.

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